Book: Bifurcation Analysis of Fluid Flows
Authors: Fred W. Wubs and Henk A. Dijkstra
Chapter: 4, Exercise: 4.3
Exercise author: G. Tiesinga
Version: 1

b. Consider the differential equation $\frac{d}{dx}(\cos(x)\frac{du}{dx}) - 3\frac{d(xu)}{dx} + \tanh(x/10)u = \exp(x)$ on $[0,1]$ with $u(0) = 1$ and $u(1) = -2$.

i. write the differentieal equation as $\frac{d}{dx}(\cos(x)\frac{du}{dx} - 3xu) + \tanh(x/10)u = \exp(x)$ integrate over control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and compute or approximate integrals (note since $\tanh(x/10)u$ is not in the form $\frac{d}{dx}\ldots$ its integral is approximated in the same way as that of the right hand side)

$$(\cos(x)\frac{du}{dx}) - 3xu)|_{i+\frac{1}{2}} - (\cos(x)\frac{du}{dx} - 3xu)|_{i-\frac{1}{2}} + \tanh(x_i/10)u_i h = \exp(x_i)h$$

ii. approximate at volume interfaces

$$(\cos(x_{i+\frac{1}{2}})\frac{u_{i+1}-u_i}{h} - 3x_{i+\frac{1}{2}}\frac{u_{i+1}+u_i}{2}) - (\cos(x_{i-\frac{1}{2}})\frac{u_i-u_{i-1}}{h} - 3x_{i-\frac{1}{2}}\frac{u_i+u_{i-1}}{2})$$
$$+ h\tanh(x_i/10)u_i = \exp(x_i)h$$

Note, this can be written as:

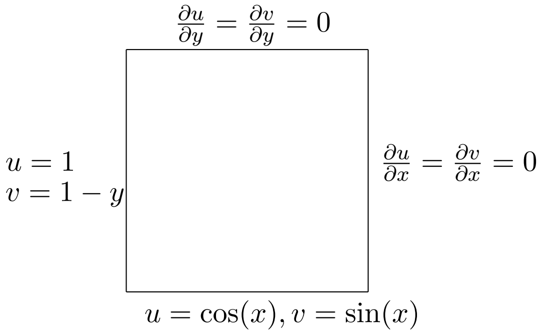
$$(\frac{\cos(x_{i+\frac{1}{2}})}{h} - \frac{3}{2}x_{i+\frac{1}{2}})u_{i+1} + (\frac{-\cos(x_{i+\frac{1}{2}})}{h} - \frac{3}{2}x_{i+\frac{1}{2}} - \frac{\cos(x_{i-\frac{1}{2}})}{h} - \frac{3}{2}x_{i-\frac{1}{2}} + h\tanh(x_i/10))u_i$$
$$+ (\frac{\cos(x_{i-\frac{1}{2}})}{h} + \frac{3}{2}x_{i-\frac{1}{2}})u_{i-1} = \exp(x_i)h$$

iii. use boundary condition:
if boundary at grid points: $u_0 = u(0) = 1$ and $u_n = u(1) = -2$
Equation derived in 2. holds for $i = 1\ldots n-1$.
In equation for $i = 1$ substitute $u_0 = 1$
In equation for $i = n-1$ substitute $u_n = -2$

iv. use boundary condition:
if boundary at volume interface, i.e. between grid points:
Equation derived in 2. holds for $i = 1\ldots n$
$x_0 = -\frac{1}{2}h$ and $x_{n+1} = 1 + \frac{1}{2}h$ fictive points
Since $u(0) = 1 \Rightarrow \frac{u_0+u_1}{2} = 1 \Rightarrow u_0 = 2 - u_1$, substitute this in equation for $i = 1$ for u_0.
Since $u(1) = -2 \Rightarrow \frac{u_n+u_{n+1}}{2} = -2 \Rightarrow u_{n+1} = -4 - u_n$ substitute this in equation for $i = n$ for u_{n+1}.

d Consider the 2D convection-diffusion equations

$$-\frac{\partial u^2}{\partial x} - \frac{\partial uv}{\partial y} + \mu\Delta u = 0$$
$$-\frac{\partial uv}{\partial x} - \frac{\partial v^2}{\partial y} + \mu\Delta v = 0$$

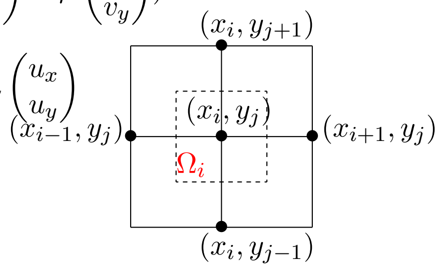


on a square $[0,1]\times[0,1]$ with boundary conditions as depicted.
Give the finite volume discretization of this problem on an equidistant grid equal in both directions. The boundary conditions should be imposed at the interfaces of control volumes. The treatment should be in line with how it is done for the 1D Burgers equation.

i. write in divergence form

$$\text{div}(\begin{pmatrix}-u^2\\-uv\end{pmatrix} + \mu\begin{pmatrix}u_x\\u_y\end{pmatrix}) = 0, \quad \text{div}(\begin{pmatrix}-uv\\-v^2\end{pmatrix} + \mu\begin{pmatrix}v_x\\v_y\end{pmatrix}) = 0$$

ii. First consider flux $\vec{q}(x,y) = \begin{pmatrix}q_1(x,y)\\q_2(x,y)\end{pmatrix} = \begin{pmatrix}-u^2\\-uv\end{pmatrix} + \mu\begin{pmatrix}u_x\\u_y\end{pmatrix}$



We get $\int_{\Omega_i} \text{div}\,\vec{q}\,d\Omega = 0 \Rightarrow \int_\Gamma (\vec{q},\vec{n})\,d\Gamma = 0$

Hence, with Ω_i as in picture with equidistant grid

$$\int_{x_{i-\frac12}}^{x_{i+\frac12}} (\vec{q}, \begin{pmatrix}0\\-1\end{pmatrix})\big|_{y=y_{j-\frac12}}\,dx + \int_{x_{i-\frac12}}^{x_{i+\frac12}} (\vec{q}, \begin{pmatrix}0\\1\end{pmatrix})\big|_{y=y_{j+\frac12}}\,dx$$
$$+\int_{y_{j-\frac12}}^{y_{j+\frac12}} (\vec{q}, \begin{pmatrix}-1\\0\end{pmatrix})\big|_{x=x_{i-\frac12}}\,dy + \int_{y_{j-\frac12}}^{y_{j+\frac12}} (\vec{q}, \begin{pmatrix}1\\0\end{pmatrix})\big|_{x=x_{i+\frac12}}\,dy = 0$$
$$\Rightarrow \int_{x_{i-\frac12}}^{x_{i+\frac12}} (q_2(x, y_{j+\frac12}) - q_2(x, y_{j-\frac12}))dx + \int_{y_{j-\frac12}}^{y_{j+\frac12}} (q_1(x_{i+\frac12}, y) - q_1(x_{i-\frac12}, y))dy = 0$$

Applying midpoint rule to integrals gives

$$(q_2(x_i, y_{j+\frac12}) - q_2(x_i, y_{j-\frac12}))\,h + (q_1(x_{i+\frac12}, y_j) - q_1(x_{i-\frac12}, y_j))\,h = 0$$

where

$$q_1(x_{i+\frac12}, y_j) = (-u^2 + \mu u_x)\big|_{i+\frac12 j} \;\Rightarrow\; (q_1)_{i+\frac12 j} = -(\frac{u_{i+1j}+u_{ij}}{2})^2 + \mu\frac{u_{i+1j}-u_{ij}}{h}$$
$$q_2(x_i, y_{j+\frac12}) = (-uv + \mu u_y)\big|_{ij+\frac12} \;\Rightarrow\; (q_2)_{ij+\frac12} = -(\frac{u_{ij+1}+u_{ij}}{2}\,\frac{v_{ij+1}+v_{ij}}{2}) + \mu\frac{u_{ij+1}-u_{ij}}{h}$$

$$(q_2(x_i, y_{j+\frac{1}{2}}) - q_2(x_i, y_{j-\frac{1}{2}}))\, h + (q_1(x_{i+\frac{1}{2}}, y_j) - q_1(x_{i-\frac{1}{2}}, y_j))\, h = 0$$

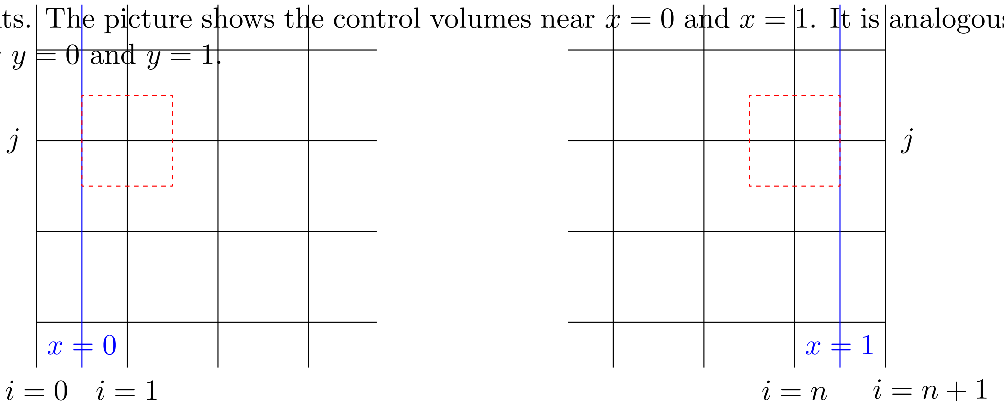
iii. An anologous approach for the other equation,
where
i.e. with $\vec{q}(x,y) = \begin{pmatrix} q_1(x,y) \\ q_2(x,y) \end{pmatrix} = \begin{pmatrix} -uv \\ -v^2 \end{pmatrix} + \mu \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

$$q_1(x_{i+\frac{1}{2}}, y_j) = (-uv + \mu v_x)\big|_{i+\frac{1}{2}j} \Rightarrow (q_1)_{i+\frac{1}{2}j} = -(\frac{u_{i+1j} + u_{ij}}{2} \frac{v_{i+1j} + v_{ij}}{2}) + \mu \frac{v_{i+1j} - v_{ij}}{h}$$

gives

$$q_2(x_i, y_{j+\frac{1}{2}}) = (-v^2 + \mu v_y)\big|_{ij+\frac{1}{2}} \Rightarrow (q_2)_{ij+\frac{1}{2}} = -(\frac{v_{ij+1} + v_{ij}}{2})^2 + \mu \frac{v_{ij+1} - v_{ij}}{h}$$

iv. Boundary conditions: assume boundary at interface volume, i.e. in between grid points. The picture shows the control volumes near $x = 0$ and $x = 1$. It is analogous near $y = 0$ and $y = 1$.



Consider the equations from part ii.
Boundary conditions at $x = 0$ and $x = 1$ are only relevant for q_1, for q_2 the boundary conditions at $y = 0$ and $y = 1$ are relevant.
Consider boundary $x = 0$. For $i = 1$, i.e. for the control volume around (x_1, y_j):

$$(q_1)_{\frac{3}{2}j} - (q_1)_{\frac{1}{2}j} = -(\frac{u_{2j} + u_{1j}}{2})^2 - (-(\frac{u_{1j} + u_{0j}}{2})^2) + \mu \frac{u_{0j} - 2u_{1j} + u_{1j}}{h} = 0$$

From the boundary condition $u(0, y) = 1 \Rightarrow \frac{u_{0j}+u_{1j}}{2} = 1 \Rightarrow u_{0j} = 2 - u_{1j}$ Substitute $u_{0j} = 2 - u_{1j}$ in the above equation (part of which will then go to the right hand side of the above equation).

Consider boundary $x = 1$. For $i = n$, i.e. for the control volume around (x_n, y_j) we have

$$(q_1)_{n+\frac{1}{2}j} - (q_1)_{n-\frac{1}{2}j} = -(\frac{u_{n+1j} + u_{nj}}{2})^2 - (-(\frac{u_{nj} + u_{n-1j}}{2})^2) + \mu \frac{u_{n+1j} - 2u_{nj} + u_{n-1j}}{h} = 0$$

For the first two terms we do not want to use the boundary condition, because for this term the boundary condition at $x = 0$ suffices. Therefor, we replace these terms by a one-sided difference, this gives

$$(q_1)_{n+\frac{1}{2}j} - (q_1)_{n-\frac{1}{2}j} = 2\big[- u_{nj}^2 - (-(\frac{u_{nj} + u_{n-1j}}{2})^2)\big] + \mu\frac{u_{n+1j} - 2u_{nj} + u_{n-1j}}{h} = 0$$

In the last term, the diffusion term, we will use the boundary condition $\frac{\partial u}{\partial x}(1, y) = 0 \Rightarrow \frac{u_{n+1j} - u_{nj}}{h} = 0$, hence we can substitute $u_{n+1j} = u_{nj}$.

v. Consider again the equations from part ii., but now the equation for q_2
For q_2 the boundary conditions at $y = 0$ and $y = 1$ are relevant. For $j = 1$, i.e. for the control volume around (x_i, y_1):

$$(q_2)_{i\frac{3}{2}} - (q_2)_{i\frac{1}{2}} = -(\frac{u_{i2} + u_{i1}}{2}\frac{v_{i2} + v_{i1}}{2}) - (-\frac{u_{i1} + u_{i0}}{2}\frac{v_{i1} + v_{i0}}{2}) + \mu\frac{u_{i2} - 2u_{i1} + u_{i0}}{h} = 0$$

From the boundary conditions: $u(x, 0) = \cos(x) \Rightarrow \frac{u_{i0}+u_{i1}}{2} = \cos(ih) \Rightarrow u_{i0} = 2\cos(ih) - u_{i1}$ and $v(x, 0) = \sin(x) \Rightarrow \frac{v_{i0}+v_{i1}}{2} = \sin(ih) \Rightarrow v_{i0} = 2\sin(ih) - v_{i1}$
Substitute $u_{i0} = 2\cos(ih) - u_{i1}$ and $v_{i0} = 2\sin(ih) - v_{i1}$ (which will then partly go to the right hand side of the above equation) in the above equation.

Consider boundary $y = 1$. For $j = n$, i.e. for the control volume around (x_i, y_n) we have

$$(q_2)_{in+\frac{1}{2}} - (q_2)_{in-\frac{1}{2}} = -(\frac{u_{in+1} + u_{in}}{2}\frac{v_{in+1} + v_{in}}{2}) - (-\frac{u_{in} + u_{in-1}}{2}\frac{v_{in} + v_{in-1}}{2}) + \mu\frac{u_{in+1} - 2u_{in} + u_{in-1}}{h} = 0$$

For the first two terms we do not want to use the boundary condition, because for this term the boundary condition at $y = 0$ suffices. Therefor, we replace these terms by a one-sided difference, this gives

$$(q_2)_{in+\frac{1}{2}} - (q_2)_{in-\frac{1}{2}} = 2\big[- u_{in}v_{in} - (-\frac{u_{nj} + u_{n-1j}}{2}\frac{v_{nj} + v_{n-1j}}{2})\big] + \mu\frac{u_{in+1} - 2u_{in} + u_{in-1}}{h} = 0$$

In the last term, the diffusion term, we will use the boundary condition $\frac{\partial u}{\partial y}(x, 1) = 0 \Rightarrow \frac{u_{in+1} - u_{in}}{h} = 0$, hence we can substitute $u_{in+1} = u_{in}$.

vi. analogous approach for the equations of part iii.