Book: Bifurcation Analysis of Fluid Flows
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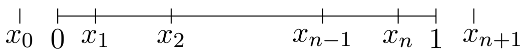
Grid 1: $x_i = ih$ for $i = 0 \dots n$ with $h = 1/n$

Grid 2: $x_i = (i - \frac{1}{2})h$ for $i = 1 \dots n$ with $h = 1/n$



here x_0 and x_{n+1} are fictive points

a.

1. second-order finite difference scheme:

$$-\cos(x_i)\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - 3x_i\frac{u_{i+1} - u_{i-1}}{2h} + \tanh(\frac{x_i}{10})u_i = e^{x_i}$$

   for internal grid points:
   Grid 1: for $i = 1 \dots n-1$
   Grid 2: for $i = 1 \dots n$

2. Boundary conditions:
   Grid 1: $u_0 = 1$ and $u_n = -2$
   use this in the finite difference scheme when $i = 1$ and $i = n-1$ respectively

   Grid 2: $\frac{u_1+u_0}{2} = 1 \;\Rightarrow\; u_0 = 2 - u_1$,
   use this expression for u_0 in the finite difference scheme when $i = 1$

   $\frac{u_n+u_{n+1}}{2} = -2 \;\Rightarrow\; u_{n+1} = -4 - u_n$,
   use this expression for u_{n+1} in the finite difference scheme when $i = n$

b.

1. Discretization of first term in two steps, first central difference discretization (hence second-order) of first order derivative using $i + \frac{1}{2}$ and $i - \frac{1}{2}$.
   First step:

$$-\Bigg(\frac{\cos(x_{i+\frac{1}{2}})\frac{du}{dx}\Big|_{i+\frac{1}{2}} - \cos(x_{i-\frac{1}{2}})\frac{du}{dx}\Big|_{i-\frac{1}{2}}}{h}\Bigg) - 3\,\frac{x_{i+1}u_{i+1} - x_{i-1}u_{i-1}}{2h} + \tanh(\frac{x_i}{10})u_i = e^{x_i}$$

   Second step gives final finite difference scheme:

$$-\Bigg(\frac{\cos(x_{i+\frac{1}{2}})\frac{u_{i+1}-u_i}{h} - \cos(x_{i-\frac{1}{2}})\frac{u_i-u_{i-1}}{h}}{h}\Bigg) - 3\,\frac{x_{i+1}u_{i+1} - x_{i-1}u_{i-1}}{2h} + \tanh(\frac{x_i}{10})u_i = e^{x_i}$$

for internal grid points:
Grid 1: for $i = 1 \dots n-1$
Grid 2: for $i = 1 \dots n$

2. boundary conditions as in part a.

c. We assume a grid in both x- and y- direction with grid size h, with rid points (x_i, y_j) given by
Grid 1: $x_i = ih$ for $i = 0 \dots n$, $y_j = jh$ for $j = 0 \dots n$ with $h = 1/n$
Grid 2: $x_i = (i - \frac{1}{2})h$ for $i = 1 \dots n$, $y_j = (j - \frac{1}{2})h$ for $j = 1 \dots n$ with $h = 1/n$.

For your own understanding, draw a square with grid and position of grid points for both Grid 1 and Grid 2 (analogously to the drawings of the two types of the 1D grid given at the beginning of the exercise)

1. The separate terms give:

$$-\frac{\partial}{\partial x}(u) \Rightarrow -\frac{u_{i+1j} - u_{i-1j}}{2h}$$

$$-\frac{\partial}{\partial y}((1+x)u)) \Rightarrow -\frac{(1+x_i)u_{ij+1} - (1+x_i)u_{ij-1}}{2h}$$

$$+\frac{\partial}{\partial x}((1+xy)\frac{\partial u}{\partial x}) \Rightarrow +\frac{(1+x_{i+\frac{1}{2}}y_j)\frac{\partial u}{\partial x}\Big|_{i+\frac{1}{2}j} - (1+x_{i-\frac{1}{2}}y_j)\frac{\partial u}{\partial x}\Big|_{i-\frac{1}{2}j}}{h}$$

$$\Rightarrow +\frac{(1+x_{i+\frac{1}{2}}y_j)\frac{u_{i+1j}-u_{ij}}{h} - (1+x_{i-\frac{1}{2}}y_j)\frac{u_{ij}-u_{i-1j}}{h}}{h}$$

$$-\frac{1}{2}\frac{\partial}{\partial x}((1+xy)\frac{\partial u}{\partial y}) \Rightarrow -\frac{1}{2}\frac{(1+x_{i+1}y)\frac{\partial u_{i+1}(y)}{\partial y} - (1+x_{i-1}y)\frac{\partial u_{i-1}(y)}{\partial y}}{2h}$$

$$\Rightarrow -\frac{1}{2}\frac{(1+x_{i+1}y_j)\frac{u_{i+1j+1}-u_{i+1j-1}}{2h} - (1+x_{i-1}y_j)\frac{u_{i-1j+1}-u_{i-1j-1}}{2h}}{2h}$$

$$-\frac{1}{2}\frac{\partial}{\partial y}((1+xy)\frac{\partial u}{\partial x}) \Rightarrow -\frac{1}{2}\frac{(1+xy_{j+1})\frac{\partial u_{j+1}(x)}{\partial x} - (1+xy_{j-1})\frac{\partial u_{j-1}(x)}{\partial x}}{2h}$$

$$\Rightarrow -\frac{1}{2}\frac{(1+x_iy_{j+1})\frac{u_{i+1j+1}-u_{i-1j+1}}{2h} - (1+x_iy_{j-1})\frac{u_{i+1j-1}-u_{i-1j-1}}{2h}}{2h}$$

$$+\frac{\partial}{\partial y}((1+xy)\frac{\partial u}{\partial y}) \Rightarrow +\frac{(1+x_iy_{j+\frac{1}{2}})\frac{\partial u_i(y)}{\partial y}\Big|_{j+\frac{1}{2}} - (1+x_iy_{j-\frac{1}{2}})\frac{\partial u_i(y)}{\partial y}\Big|_{j-\frac{1}{2}}}{h}$$

$$\Rightarrow +\frac{(1+x_iy_{j+\frac{1}{2}})\frac{u_{ij+1}-u_{ij}}{h} - (1+x_iy_{j-\frac{1}{2}})\frac{u_{ij}-u_{ij-1}}{h}}{h}$$

for internal grid points:
Grid 1: for $i = 1 \dots n-1$ and $j = 1 \dots n-1$
Grid 2: for $i = 1 \dots n$ and $j = 1 \dots n$

2. Boundary conditions
For your own understanding you can add the continuous boundary conditions to the sides of the squares showing the grid and grid points of Grid 1 and Grid 2, you drew before.

Grid 1:

$u(0,y) = 1 \Rightarrow u_{0j} = 1$
use this in the finite difference scheme when $i = 1$

$u(x,0) = \cos(x) \;\Rightarrow\; u_{i0} = \cos(x_i)$

use this in the finite difference scheme when $j = 1$

because u is not known at $x = 1$ and $y = 1$ (due to the Neumann boundary conditions), we need to compute u_{nj} and u_{in} as well

$\frac{du}{dx}(1,y) = 0 \;\Rightarrow\; \frac{u_{n+1j} - u_{n-1j}}{2h} = 0 \;\Rightarrow\; u_{n+1j} = u_{n-1j}$

we introduced the fictive points x_{n+1j} to obtain second order accuracy in treatment

of boundary condition and solution

use the finite difference scheme given in 1. for $i = n$ as well, and use the just derived formula for u_{n+1j} in the scheme for $i = n$

$\frac{du}{dy}(x,1) = 0 \;\Rightarrow\; \frac{u_{in+1} - u_{in-1}}{2h} = 0 \;\Rightarrow\; u_{in+1} = u_{in-1}$

we introduced the fictive points x_{in+1} to obtain second order accuracy in treatment

of boundary condition and solution

use the finite difference scheme given in 1. for $j = n$ as well, and use the just derived formula for u_{in+1} in the scheme for $j = n$

Grid 2:

$u(0,y) = 1 \;\Rightarrow\; \frac{u_{0j} + u_{1j}}{2} = 1 \;\Rightarrow\; u_{0j} = 2 - u_{1j}$

use this formula for u_{0j} in the finite difference scheme given in 1. when $i = 1$

$u(x,0) = \cos(x) \;\Rightarrow\; \frac{u_{i0} + u_{i1}}{2} = \cos(x_i) \;\Rightarrow\; u_{i0} = 2\cos(x_i) - u_{i1}$

use this formula for u_{i0} in the finite difference scheme given in 1. when $j = 1$

$\frac{du}{dx}(1,y) = 0 \;\Rightarrow\; \frac{u_{n+1j} - u_{nj}}{h} = 0 \;\Rightarrow\; u_{n+1j} = u_{nj}$

use this formula for u_{n+1j} in the finite difference scheme given in 1. when $i = n$

$\frac{du}{dy}(x,1) = 0 \;\Rightarrow\; \frac{u_{in+1} - u_{in}}{h} = 0 \;\Rightarrow\; u_{in+1} = u_{in}$

use this formula for u_{in+1} in the finite difference scheme given in 1. when $j = n$

d.
1. Grid as in part c.
2. Finite difference scheme

$$-\frac{(u_{i+1j})^2 - (u_{i-1j})^2}{2h} - \frac{v_{ij+1}u_{ij+1} - v_{ij-1}u_{ij-1}}{2h} + \mu\left(\frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2}\right) = 0$$

$$-\frac{u_{i+1j}v_{i+1j} - u_{i-1j}v_{i-1j}}{2h} - \frac{(v_{ij+1})^2 - (v_{ij-1})^2}{2h} + \mu\left(\frac{v_{i+1j} - 2v_{ij} + v_{i-1j}}{h^2} + \frac{v_{ij+1} - 2v_{ij} + v_{ij-1}}{h^2}\right) = 0$$

3. Treatment of boundary conditions analogous approach as in part c.