Book: Bifurcation Analysis of Fluid Flows
Authors: Fred W. Wubs and Henk A. Dijkstra
Chapter: 4, Exercise: 4.17
Exercise author: G. Tiesinga
Version: 1

For stability of time integration method consider test equation $\frac{du}{dt} = \lambda u$

a. Forward Euler $u^{n+1} = u^n + \Delta t \lambda u^n$. Show that for Forward Euler for stability we need that $|1 + \Delta t\lambda| < 1$. If λ is purely imaginary and $\lambda \neq 0$ we have $|1 + \Delta t i Im(\lambda)| = \sqrt{1 + (\Delta t Im(\lambda))^2} > 1$ for all $\Delta t > 0$. Hence, method unstable for problems with only purely imaginary eigenvalues ($u^n = (1 + \Delta\lambda)^n u_0$ will not go to zero for $n \Rightarrow \infty$).
Note: setting $z = \Delta t\lambda$, we need $|1 + z| < 1$, i.e z (complex number, i.e. $x + iy$), z inside disk with center $(x, y) = (-1, 0)$ and radius 1.

b. Backward Euler: $u^{n+1} = u^n + \Delta t\lambda u^{n+1}$. Hence $(1 - \Delta t\lambda)u^{n+1} = u^n$ and method stable if $|\frac{1}{1-\Delta t\lambda}| < 1$. Setting $z = \Delta t\lambda$, we see that we need $|1 - z| > 1$. Hence z (complex number, i.e. $x + iy$), outside disk with center $(x, y) = (1, 0)$ and radius 1).

c. Note: method is A-stable if the region of absolute stability contains the half plain $Re(z) < 0$.
Backward Euler and Trapezoidal method/Crank-Nicolson method have $Re(z) < 0$ in their region of absolute stability, and are hence A-stable.
BDF(k) is A-stable only for $k = 1$ (backward Euler) and $k = 2$.
(see for instance https://en.wikipedia.org/wiki/Backward_differentiation_formula where the pink regions are the regions of stability)

For

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(t, x) \quad a > 0,\ t > 0,\ 0 < x < 1$$
$$u(x, 0) = g(x),\quad u(0, t) = u(1, t) = 0$$

we get

$$\frac{dU}{dt} = AU + \hat{f},\quad U(0) = \hat{g}$$

where A is given by formula (2.69) of the reader and its eigenvalues are real and in the interval $[-4a/h^2, 0)$ (see formula (2.77) and below in the Lecture Notes). From the regions of stability of the BDF(k) methods, we see that all these methods are stable for $z = \Delta t\lambda$ real and negative. Hence, non of the BDF(k) methods for the time integration, will give a restriction on the time time Δt

d. Trapezoidal method/Crank-Nicolson method: $u^{n+1} = u^n + \frac{\Delta t}{2}\left[\lambda u^n + \lambda u^{n+1}\right]$. Show that this can be written as $u^{n+1} = \frac{1+\frac{1}{2}\Delta t\lambda}{1-\frac{1}{2}\Delta t\lambda}\, u^n$. Hence, we need that the amplification factor satisfies $|\rho(z)| = |\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}| < 1$ and hence $|2-z| > |2+z|$ which gives $Re(z) < 0$.

d. The amplification factor $\rho(z)$ of the trapezoidal method for $|z| \;\rightarrow\; \infty$ is

$$\lim_{|z| \;\rightarrow\; \infty} \frac{1+\frac{1}{2}z}{1-\frac{1}{2}z} = \lim_{|z| \;\rightarrow\; \infty} \frac{\frac{1}{z}+\frac{1}{2}}{\frac{1}{z}-\frac{1}{2}} = -1.$$

Note: For the test equation we have $\frac{du}{dt} = \lambda u \;\Rightarrow\; u(t) = \exp(\lambda t)u(0) \;\Rightarrow\; u(t_{n+1}) = u(t_n + \Delta t) = \exp(\lambda(t_n + \Delta t))u(t_n) = \exp(z)u(t_n)$. Hence for the continuous solution we have amplification factor e^z. We have $\exp(z) = \exp(Re(z))\exp(iIm(z)$ and see that the behaviour of this amplification factor for $|z| \;\rightarrow\; \infty$ is completely different of that of the trapezoidal method.