Book: Bifurcation Analysis of Fluid Flows
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Apply θ-method to $M\frac{d}{dt}c(t) = Ac(t) + \hat{f}(t)$, $c(0) = c_0$, where $c(t)$ is a vector (equation 2.72 of the lecture notes)

Note: This ODE stems from applying the FEM to $\frac{\partial}{\partial t}u(t,x) = \mu\frac{\partial^2}{\partial t^2}u(t,x) + f(t,x)$, i.e. using the Galerkin approach and setting $u(t,x) = \sum c_i(t)\phi_i(x)$. The matrix A does not depend on t and is given by $A_{ij} = \mu(\phi_i'(x), \phi_j'(x)) = \mu\int_0^1 \phi_i'(x), \phi_j'(x)dx$. Furthermore, $\hat{f}$ is given by $\hat{f}_i = (f(t,x), \phi_i(x)) = \int_0^1 f(t,x)\phi_i(x)dx$.

Applying the θ-method gives

$$
\begin{aligned}
Mc^{(n+1)} &= Mc^{(n)} + \Delta t\big[(1-\theta)(Ac^{(n)} + \hat{f}(t_n)) + \theta(Ac^{(n+1)} + \hat{f}(t_{n+1}))\big] \\
\Rightarrow (M - \theta\Delta t A)c^{(n+1)} &= Mc^{(n)} + \Delta t\big[(1-\theta)Ac^{(n)} + (1-\theta)\hat{f}(t_n) + \theta\hat{f}(t_{n+1})\big]
\end{aligned}
$$

with $c^{(0)} = c_0$.