Book: Bifurcation Analysis of Fluid Flows
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a. Consider $\frac{\partial}{\partial t}u(x,t) = -\frac{\partial}{\partial x}u(x,t)$ for $x \in [0,1]$ and $t > 0$ with boundary condition $u(0,t) = 1$ and initial condition $u(x,0) = 0$.
We use equidistant grid with grid size $h = \frac{1}{m}$ and grid points coincide with boundary, i.e. $x_0 = 0$ and $x_m = 1$.
We have unknown in x_n since there is no boundary condition at $x = 1$. We use backward difference discretization at $x = 1$.

1. Space discretization (2.4), i.e. central difference, for internal points gives following ODEs

$$\frac{du_j}{dt} = -\frac{u_{j+1} - u_{j-1}}{2h} \qquad j = 1 \ldots m-1$$
$$\frac{du_m}{dt} = -\frac{u_m - u_{m-1}}{h}$$

2. Space discretization (2.6), i.e. backward difference, for internal points gives following ODEs

$$\frac{du_j}{dt} = -\frac{u_j - u_{j-1}}{h} \qquad j = 1 \ldots m$$

3. Space discretization (2.5), i.e. forward difference, for internal points gives following ODEs

$$\frac{du_j}{dt} = -\frac{u_{j+1} - u_j}{h} \qquad j = 1 \ldots m-1$$
$$\frac{du_m}{dt} = -\frac{u_m - u_{m-1}}{h}$$

Note: in the above u_j for $j = 1 \ldots m$, depend on t, i.e. $u_j(t)$.

In all cases:
initital condition $u_j(0) = 0$ for $j = 1 \ldots m$.
boundary condition $u_0(t) = 1$ for $t > 0$.

In case of forward discretization, the discretization does not use u_0, i.e. it does not use the boundary condition $u_0(t) = 1$. Hence, because $u_j(0) = 0$ for $j = 1 \ldots m$ the solution will be 0 for all time (i.e. $u_j(t) = 0$ for all t and for $j = 1 \ldots m$.
Conclusion: this discetization is not suited for this problem.

b. Use the difference/Fourier method on the three discretizations in part a.
Note:
- this analysis is only applied to the discretization of the internal grid points.
- this analysis computes the eigenvalues of the space discretization belonging to the eigen function $u_j = \exp(ij\varphi)$.

Substitute $u_j = \exp(ij\varphi)$ (where $i = \sqrt{-1}$ and $-\pi < \varphi \leq \pi$) in the space discretization of the internal points, and perform the computation to show:

1. Space discretization (2.4), i.e. central difference, for internal points

$$-\frac{u_{j+1} - u_{j-1}}{2h} = -i\frac{\sin\varphi}{2h}\exp(ij\varphi)$$

Hence, $\lambda = -i\frac{\sin\varphi}{2h}$ and $Re(\lambda) = 0$.
Small perturbations in the initial solution will not increase:
the solution of the test equation $\frac{du}{dt} = \lambda u, \quad u(0) = \epsilon$ is $u(t) = \epsilon\exp(\lambda t)$. If $Re(\lambda) = 0$ then $|u(t) = |\epsilon|$, which means that $u(t)$ is bounded in the initial condition for all t hence it is stable.

Note: the ODEs resulting from the central discretization (given in part a.) can not be solved with Forward Euler, since the Forward Euler method is unstable for problems with purely imaginary eigenvalues.

2. Space discretization (2.6), i.e. backward difference, for internal points

$$-\frac{u_j - u_{j-1}}{h} = -\frac{1 - \exp(-i\varphi)}{h}\exp(ij\varphi)$$

We have $\lambda = -\frac{1-\exp(-i\varphi)}{h} = -\frac{1-\cos\varphi+i\sin\varphi}{h}$, and hence $Re(\lambda) = -\frac{1-\cos\varphi}{h} \leq 0$ and consequently the discretization results in a stable ODE. Small perturbations in the initial solution will not increase.

3. Space discretization (2.5), i.e. forward difference, for internal points

$$-\frac{u_{j+1} - u_j}{h} = -\frac{\exp(i\varphi) - 1}{h}\exp(ij\varphi)$$

We have $\lambda = -\frac{\exp(i\varphi)-1}{h} = -\frac{\cos\varphi-1+i\sin\varphi}{h}$, and hence $Re(\lambda) = -\frac{\cos\varphi-1}{h} \geq 0$ and consequently the discretization results in a non-stable ODE. Small perturbations in the initial solution will grow.
Conclusion: this space discretization is not suited for this problem