Book: Bifurcation Analysis of Fluid Flows
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Note: not 1D but 2D

1. Minimize $J(u, v, p) = J(u, v) + (p, u_x + v_u)$. Where u, v, p are all functions depending on x, y.
2. Show $\frac{d}{d\epsilon} J(u, v, p + \epsilon\tilde{p})\Big|_{\epsilon=0} = 0 \;\Rightarrow\; (\tilde{p}, u_x + v_y) = 0 \;\forall \tilde{p}$
3. Show $\frac{d}{d\epsilon} J(u + \epsilon\tilde{u}, v, p)\Big|_{\epsilon=0} = 0 \;\Rightarrow\; (\tilde{u}, -\Delta u - f_1 + p_x) = 0 \;\forall \tilde{u}$
   Use $\frac{d}{d\epsilon} J(u + \epsilon\tilde{u}, v, p)\Big|_{\epsilon=0} = 0 \;\Rightarrow \frac{1}{2}(\tilde{u}, \Delta u) + \frac{1}{2}(u, -\Delta\tilde{u}) - (f, \tilde{u}) + (p, \tilde{u}_x) = 0$. Use twice partial integration with $u = 0$ on Γ (or using equation (1.38) in the lecture notes), to show $(u, -\Delta\tilde{u}) = (\tilde{u}, -\Delta u)$. Use partial integration to show $(p, \tilde{u}_x) = (p_x, \tilde{u})$.
4. Show $\frac{d}{d\epsilon} J(u, v + \epsilon\tilde{v}, p)\Big|_{\epsilon=0} = 0 \;\Rightarrow\; (\tilde{v}, -\Delta v - f_2 + p_y) = 0 \;\forall \tilde{v}$

Hence, we obtain

$$
\begin{aligned}
&(\tilde{p}, u_x + v_y) = 0 \\
&(\tilde{u}, -\Delta u - f_1 + p_x) = 0 \\
&(\tilde{v}, -\Delta v - f_2 + p_y) = 0
\end{aligned}
$$

which is the weak form of the Stokes equations

$$
\begin{aligned}
&u_x + v_y = 0 \\
&-\Delta u + p_x = f_1 \\
&-\Delta v + p_y = f_2
\end{aligned}
$$