# Book: Bifurcation Analysis of Fluid Flows
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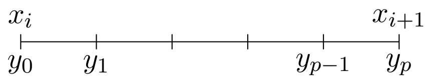
a. 1. Grid on $[0,1]$, grid size $h = \frac{1}{n}$, i.e. $x_i = ih$. Consider one element $x_i, x_{i+1}$ and consider on that element $p$-th order polynomial interpolation of $f$, i.e. $q_p(x) = (x-y_0)(x-y_1)\ldots(x-y_p)$, with $y_j = x_i + jm$ where $m = h/p$ .




2. We have on element $[x_i, x_{i+1}]$ that $f(x) = q_p(x) + (x-y_0)(x-y_1)\ldots(x-y_p)\frac{f^{p+1}(\xi(x))}{(p+1)!}$
The error of the piecewise interpolating polynomial $\Pi_{\mathcal{V}_n} f$ on $[0,1]$ is

$$\|f - \Pi_{\mathcal{V}_n} f\| = \sqrt{\sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} (f(x) - \Pi_{\mathcal{V}_n} f)^2 dx} \qquad (**)$$

3. First consider error on 1 element:

$$\int_{ih}^{(i+1)h} (f(x) - \Pi_{\mathcal{V}_n} f)^2 dx = \int_{ih}^{(i+1)h} (x-y_0)^2(x-y_1)^2\ldots(x-y_p)^2 \left(\frac{f^{p+1}(\xi(x))}{(p+1)!}\right)^2 dx$$
$$\leq \max_{x\in[x_i,x_{i+1}]}\left(\left(\frac{f^{p+1}(\xi(x))}{(p+1)!}|\right)^2\right) \int_{ih}^{(i+1)h} (x-y_0)^2(x-y_1)^2\ldots(x-y_p)^2 dx$$

Using $x = x_i + sh$, with $s \in [0,1]$, show $dx = hds$ and $(x - y_j) = (s - \frac{j}{p})h$, and

$$\int_{ih}^{(i+1)h} (x-y_0)^2(x-y_1)^2\ldots(x-y_p)^2 dx = h^{2p+3}\left(\int_0^1 \prod_{j=0}^{p}(s-\tfrac{j}{p})ds\right)^2$$

Note $\int_0^1 \prod_{j=0}^{p}(s - \frac{j}{p})ds$ does not depend on $h$

4. Now return to (**), the error over the whole interval $[0,1]$: argue that

$$\|f - \Pi_{\mathcal{V}_n} f\| \leq \sqrt{h^{2p+3}\sum_{i=0}^{n-1} \max_{x\in[x_i,x_{i+1}]}\left(\left(\frac{f^{p+1}(\xi(x))}{(p+1)!}\right)^2\right)\int_0^1 \prod_{j=0}^{p}(s-\tfrac{j}{p})^2 ds}$$
$$\leq h^{p+1}\max_{x\in[0,1]}\left(\left|\frac{f^{p+1}(\xi(x))}{(p+1)!}\right|\right)\sqrt{nh\int_0^1 \prod_{j=0}^{p}(s-\tfrac{j}{p})^2 ds}$$
$$= h^{p+1}\max_{x\in[0,1]}\left(\left|\frac{f^{p+1}(\xi(x))}{(p+1)!}\right|\right)\sqrt{c_n} = O(h^{p+1})$$

where $c_n = \int_0^1 \prod_{j=0}^{p}(s - \frac{j}{p})^2 ds$. You can show $c_n \leq \int_0^1 1ds = 1$

5. From Corollary 4.1 we know that the error of the best approximation is bounded by the error of the interpolation polynomial: $\|f - f_{\mathcal{V}}\|_{H^0} = \|f - f_{\mathcal{V}}\| \leq \frac{M}{c}\|f - \Pi_{\mathcal{V}} f\|_{H^0} = O(h^{p+1})$

b. 1. Sobolev norm $\|f\|_{H^k} = \sqrt{\|f\|^2 + \|\frac{df}{dx}\|^2 + \ldots + \|\frac{d^k f}{dx^k}\|^2}$
We have $f(x) = q_p(x) + (\prod_{j=0}^{p}(x-y_j))\frac{f^{p+1}(\xi(x))}{(p+1)!}$ and from a. $\|f\|^2 = (O(h^{p+1}))^2$
Argue $\frac{d}{dx}(f(x) - q_p(x)) = \frac{f^{p+1}(\xi(x))}{(p+1)!}\frac{d}{dx}(\prod_{j=0}^{p}(x-y_j)) + \prod_{j=0}^{p}(x-y_j)\frac{d}{dx}(\frac{f^{p+1}(\xi(x))}{(p+1)!})$

and hence $\frac{d^k}{dx^k}(f(x) - q_p(x)) = \frac{f^{p+1}(\xi(x))}{(p+1)!}\frac{d^k}{dx^k}(\prod_{j=0}^{p}(x - y_j)) + \ldots$ where we only mentioned the term which has lowest power of $h$ (i.e. worst case scenario for order of error).

2. As before, using $x = x_i + sh$, with $s \in [0, 1]$, show $dx = hds$, $\frac{d^k}{dx^k} = \frac{1}{h^k}\frac{d^k}{ds^k}$ and $\frac{d^k}{dx^k}(\prod_{j=0}^{p}(x - y_j)) = h^{p+1-k}\frac{d^k}{ds^k}\prod_{j=0}^{p}(s - \frac{j}{p})$ and argue analogously as in part a. by considering the error first at one element and then at the whole interval $[0, 1]$ to obtain $\|f - \Pi_{\mathcal{V}} f\|_{H^k} = O(h^{p+1-k})$
3. Finally argue that $\|f - f_{\mathcal{V}}\|_{H^k} = O(h^{p+1-k})$