Book: Bifurcation Analysis of Fluid Flows
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Apply FEM with piecewise linear base functions to the cases of Exercise 3.15a. For application to 3.15 a.1 we have currently only a handwritten document inserted at the end of the document. (Note that Exc. 2.11 should be Exc 4.11 and 1.15 should be 3.15.) This provides us with a matrix A and a right-hand side b. For the case below, we will indicate how A and b need to be adjusted.

a.2
1. matrix A does not change
2. the right hand side b changes: instead of computing each time (ϕ, x^2) we need to compute $(\phi, x^2 - (1+x)(1+4))$
3. at the end $\tilde{u} = \sum_{i=1}^{n-1} c_i\phi_i(x)$ and $u = 1 + 4x + \sum_{i=1}^{n-1} c_i\phi_i(x)$

a.3
1. We have: $a(v,u)$ as before. But now, $u(0)$ is not known. Hence, also use ϕ_0 and c_0, i.e. $u(x) = \sum_{i=0}^{n-1} c_i\phi_i(x)$. The size of the matrix A will not be $(n-1)\times(n-1)$ as in part a.1 but $n \times n$. It will have one additional row and one additional column at the beginning (the rest of the matrix is as before), containing (ϕ_0, ϕ_j) for $j = 0 \ldots n-1$ and (ϕ_i, ϕ_0) for $i = 0 \ldots n-1$. Most of these elements, as before, will be 0.
2. We have: $F(v) = (v, x^2) - 3v(0)$. Analogously to A, we now obtain b of length n instead of $n-1$. The additional element is at the top of vector b, namely the element (ϕ_0, x^2).
3. In addition, from the in this way obtained vector b we should substract 3v(0), but since $b_i = F(\phi_i)$ and $\phi_0(0) = 1$ but for all others we have $\phi_i(0) = 0$, we need to substract 3 only from the first element of b.

a.4 as in a.3, but for b use $F(v) = (v, x^2 - 5(1+x)) - 3v(0)$, i.e. in the inner product computations take $x^2 - 5(1+x)$ and not x^2

a.5 as in a.3, but now we have in $a(v,u)$ an additional term $-9v(0)u(0)$, which only changes $a(\phi_0, \phi_0)$ since $\phi_0(0) = 1$ but for all others we have $\phi_i(0) = 0$. Hence substract 9 from the first diagonal element of $a(v,u)$

a.6 see a.5 but now we have in $a(v,u)$ an additional term $+9v(0)u(0)$.

Exc 2.11

consider 1.15 a(i) $\quad a(v,u) = \left(\frac{dv}{dx}, \frac{du}{dx}\right) + (v, (1+x)u)$

$$F(v) = (v, x^2)$$

$$V = \{ v \in H^1[0,1] \mid v(0) = v(1) = 0 \}$$

use : shape function degree 1 $\quad x_0 = 0,\ x_n = 1 \quad h = \frac{1}{n}$

$x_i = ih$

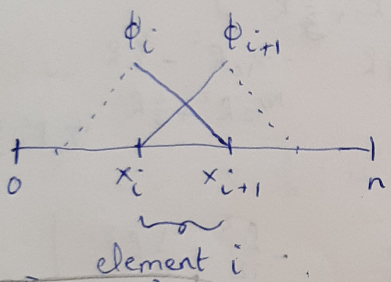
$A\vec{c} = b \qquad u(x) = \sum_{i=1}^{n-1} c_i \phi_i(x)$

$\phi_i(x)$ linear piecewise

$$\phi_i(x_j) = 1 \quad i = j$$
$$= 0 \quad i \neq j$$

$$A_{ij} = a(\phi_i, \phi_j)$$

element matrix $\begin{bmatrix} (\phi_i', \phi_i')_i & (\phi_i', \phi_{i+1}')_i \\ (\phi_{i+1}', \phi_i')_i & (\phi_{i+1}', \phi_{i+1}')_i \end{bmatrix}$



$$+ \begin{bmatrix} (\phi_i, (1+x)\phi_i)_i & (\phi_i, (1+x)\phi_{i+1})_i \\ (\phi_{i+1}, (1+x)\phi_i)_i & (\phi_{i+1}, (1+x)\phi_{i+1})_i \end{bmatrix}$$

$\boxed{\text{on } [x_i, x_{i+1}]}$:

$\phi_{i+1}\Big|_{\theta} = \frac{1}{x_{i+1} - x_i}(x - x_i) = \frac{1}{h}(x - x_i) \qquad \phi_{i+1}' = \frac{1}{h}$

$\phi_i\Big|_{\theta} = \frac{1}{x_{i+1} - x_i}(x_{i+1} - x) = \frac{1}{h}(x_{i+1} - x) \qquad \phi_i' = -\frac{1}{h}$

$$(\phi_i', \phi_i')_i = \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)^2 dx = \frac{1}{h}$$

$$(\phi_{i+1}', \phi_i')_i = \int_{x_i}^{x_{i+1}} \frac{1}{h} \cdot \frac{-1}{h}\, dx = -\frac{1}{h}$$

$$(\phi_i, (1+x)\phi_i)_i = \int_{x_i}^{x_{i+1}} \frac{1}{h}(x_{i+1} - x)(1+x) \cdot \frac{1}{h}(x_{i+1} - x)\, dx = \ldots\ldots$$

$$= +\frac{1}{h^2} \cdot +\frac{1}{12} h^3 (3x_i + x_{i+1} + 4)$$

$+\frac{1}{12}h(4ih + h + 4) = +\frac{4i+1}{12}h^2 + \frac{1}{3}h \quad = +\frac{1}{12}h(3ih + (i+1)h + 4)$

$$(\phi_{i+1},(1+x)\phi_{i+1})_i = \int_{x_i}^{x_{i+1}} \frac{1}{h^2}(x-x_i)(1+x)(x-x_i)\,dx$$

$$= \frac{1}{h^2}\,\frac{1}{12}h^3(4ih+3h+4) = \frac{h^2}{12}(4i+3)+\frac{1}{3}h$$

$$(\phi_{i+1},(1+x)\phi_i)_i = \int_{x_i}^{x_{i+1}} -\frac{1}{h^2}(x-x_i)(1+x)(x_{i+1}-x)\,dx$$

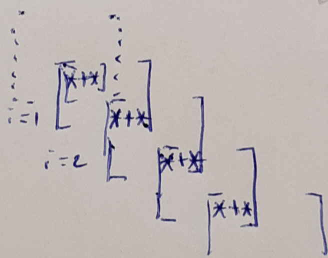
$$= -\frac{1}{h^2}\,\frac{h^3}{12}(2hi+h+2) = -\frac{h^2}{12}(2i+1)-\frac{1}{6}h$$

$$\Rightarrow\ i=0 \quad \frac{1}{h}\begin{bmatrix}1 & -1\\ -1 & \boxed{1}\end{bmatrix} + \begin{bmatrix}h^2/12+h/3 & -h^2/12-h/6\\ -h^2/12-\frac{1}{6}h & \boxed{3h^2/12+\frac{1}{3}h}\end{bmatrix}$$

$$i=1 \quad \frac{1}{h}\begin{bmatrix}\boxed{1} & -1\\ -1 & \boxed{1}\end{bmatrix} + \begin{bmatrix}\boxed{5/12\,h^2+h/3} & -3/12\,h^2-1/6\,h\\ -3/12\,h^2-h/6 & \boxed{7/12\,h^2+1/3\,h}\end{bmatrix}$$

$$i=2 \quad 1/h\begin{bmatrix}\boxed{1} & -1\\ -1 & \boxed{1}\end{bmatrix} + \begin{bmatrix}\boxed{9/12\,h^2+h/3} & -5/12\,h^2-1/6\,h\\ -5/12\,h^2-h/6 & \boxed{11/12\,h^2+1/3\,h}\end{bmatrix}$$

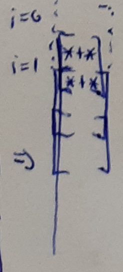
combine all elements:



$$\Rightarrow A = \frac{1}{h}\begin{bmatrix}2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1\\ & & -1 & 2\end{bmatrix} + \begin{bmatrix}8/12\,h^2+\frac{2}{3}h & -h^2/12-h/6 & & \\ -\frac{3h^2}{12}-h/6 & 16/12\,h^2+\frac{2}{3}h & -3\frac{h^2}{12}-h/6 & \\ & -\frac{5}{12}h^2-h/6 & 24/12\,h^2+\frac{2}{3}h & -\frac{5h^2}{12}-h/6\\ & & \ddots & \ddots & \ddots\end{bmatrix}$$

$$F(\phi_i)|_i = \begin{pmatrix}(\phi_i,x^2)_i\\ (\phi_{i+1},x^2)_i\end{pmatrix} = \begin{pmatrix}\int_{x_i}^{x_{i+1}} 1/h\,(x_{i+1}-x)x^2\,dx\\ \int_{x_i}^{x_{i+1}} 1/h\,(x-x_i)x^2\,dx\end{pmatrix} = \begin{pmatrix}h^2/12\,(6i^2+4i+1)\\ h^2/12\,(6i^2+8i+3)\end{pmatrix}$$

$$i=0 \begin{bmatrix}h^2/12\\ \boxed{3h^2/12}\end{bmatrix} \quad i=1 \begin{bmatrix}\boxed{h^2/12\cdot 11}\\ \boxed{h^2/12\cdot 17}\end{bmatrix} \quad i=2 \begin{bmatrix}\boxed{h^2/12\cdot 33}\\ h^2/12\cdot 43\end{bmatrix}$$



$$\Rightarrow b = \begin{bmatrix}14h^2/12\\ 50h^2/12\\ 110h^2/12\\ \vdots\end{bmatrix} \qquad \Rightarrow \text{solve } A\vec{c} = b$$

$$\Rightarrow u(x) = \sum_{i=1}^{n-1} c_i\phi_i(x)$$