Book: Bifurcation Analysis of Fluid Flows
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Chapter: 4, Exercise: 4.1
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Consider the linear convection-diffusion equation

$$-\frac{d^2u}{dx^2} - 3\frac{du}{dx} = f(x) \quad \text{on} \quad [0,1] \quad \text{with} \quad u(0) = u(1) = 1.$$

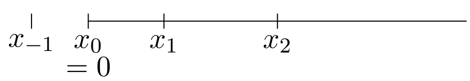
b. Let $g(\xi) = a + b\exp(\xi)$. We want a and b s.t. $g(0) = 0$ and $g(1) = 1$. Show that $a = \frac{1}{1-e}$ and $b = -\frac{1}{1-e}$.

c.
1. Non-uniform grid given by $x_i = g(\xi_i) = \frac{1}{1-e} - \frac{1}{1-e}\exp(\xi_i)$ where $\xi_i = ih,\ i = 0 \dots n$ with $h = \frac{1}{n}$
2. Use equations 2.9 and 2.10 from the Lecture Notes for the discretization on a non-uniform grid, for our differential equation we get, using $h_2 = x_{i+1} - x_i$ and $h_1 = x_i - x_{i-1}$:

$$-\left(\frac{2u_{i+1}}{h_2(h_1+h_2)} - \frac{2u_i}{h_1h_2} + \frac{2u_{i-1}}{h_1(h_2+h_1)}\right) - 3\left(\frac{h_1u_{i+1}}{h_2(h_2+h_1)} + \frac{(h_2-h_1)u_i}{h_1h_2} - \frac{h_2u_{i-1}}{h_1(h_1+h_2)}\right) = f(x_i)$$

for $i = 1 \dots n-1$ where $x_i = g(\xi_i) = g(ih) = \frac{1}{1-e} - \frac{1}{1-e}\exp(ih)$ with $h = \frac{1}{n}$, and $u_0 = u(0) = 1$ and $u_n = u(1) = 1$

d.(i) Now, assume at $x = 0$ the Neumann boundary condition $\frac{du}{dx}(0) = 2$.
With at $x = 0$ Neumann boundary condition, u_0 is now unknown, hence we need a discretization at $i = 0$ as well. We will use the discretization given in 2. also for $i = 0$ (u_{-1} is in that discretization). We will use the boundary condition to replace u_{-1}).
Second-order approximation of boundary condition at $x = 0$ requires the fictive grid point x_{-1}



Option 1
Discretizing the boundary condition $\frac{du}{dx}(0) = 2$ on non-equidistant grid gives, using equation 2.10 from the Lecture notes,

$$\frac{(x_0 - x_{-1})u_1}{(x_1 - x_0)(x_1 - x_{-1})} + \frac{(x_1 - 2x_0 + x_{-1})u_0}{(x_0 - x_{-1})(x_1 - x_0)} - \frac{(x_1 - x_0)u_{-1}}{(x_0 - x_{-1})(x_{+1} - x_{-1})} = 2$$

This equation can be re-written as $u_{-1} = \ldots$. Which can be used in the discretization given in c.2. for $i = 0$.
Option 2
The above results in a rather complicated formula for u_{-1}. A simpler form results from the following discretization of the boundary condition $\frac{du}{dx}(0) = 2$,

$$\frac{u_1 - u_{-1}}{x_1 - x_{-1}} = 2$$

This equation can be re-written as $u_{-1} = \ldots$. Which can be used in the discretization given in c.2. for $i = 0$.
Note: this discretization is the central discretization on a uniform grid, i.e. as if $x_0 = 0$ were exactly in the middle of x_{-1} and x_1 (for an explanation why this will work as well, see the note at the end of part e.)

d.(ii) Now, assume at $x = 0$ the Robin boundary condition $u(0) + 2\frac{du}{dx}(0) = 5$.
With at $x = 0$ Robin boundary condition, u_0 is now unknown, hence we need a discretization at $i = 0$ as well. We will use the discretization given in 2. also for $i = 0$ (u_{-1} is in that discretization). We will use the boundary condition to replace u_{-1}).
Similar as in the case of Neumann boundary condition, we need a fictive point x_{-1} to obtain second order accurate approximation of the boundary condition
Option 1: Discretizing the boundary condition $u(0) + 2\frac{du}{dx}(0) = 5$ on non-equidistant grid gives, using equation 2.10 from the Lecture notes,

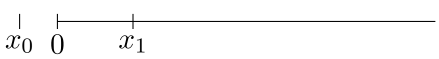
$$u_0 + 2\left(\frac{(x_0 - x_{-1})u_1}{(x_1 - x_0)(x_1 - x_{-1})} + \frac{(x_1 - 2x_0 + x_{-1})u_0}{(x_0 - x_{-1})(x_1 - x_0)} + \frac{(x_1 - x_0)u_{-1}}{(x_0 - x_{-1})(x_{+1} - x_{-1})}\right) = 5$$

This equation can be re-written as $u_{-1} = \ldots$. Which can be used in the discretization of c.2. for $i = 0$.
Option 2
The above results in a rather complicated formula for u_{-1}. A simpler form results from the following discretization of the boundary condition $u(0) + 2\frac{du}{dx}(0) = 5$,

$$u_0 + 2\frac{u_1 - u_{-1}}{x_1 - x_{-1}} = 5$$

This equation can be re-written as $u_{-1} = \ldots$. Which can be used in the discretization given in c.2. for $i = 0$.
Note: this discretization is the central discretization on a uniform grid, i.e. as if $x_0 = 0$ were exactly in the middle of x_{-1} and x_1 (for an explanation why this will work as well, see the note at the end of part e.)

e. Assume grid is such that position of left boundary is between grid points, between fictive point $x_0 = g(-h/2)$ and $x_1 = g(h/2)$.
The discretization given in c.2. still holds, but for $i = 1$ it uses the fictive u_0. Hence, we want to remove u_0 from the discretization for $i = 1$ using the boundary condition at $x = 0$.



1) Dirichlet boundary condition $u(0) = 1$: $\frac{u_0+u_1}{2} = 1 \;\Rightarrow\; u_0 = \ldots$
2) Neumann boundary condition $\frac{du}{dx}(0) = 2$: $\frac{u_1-u_0}{x_1-x_0} \;\Rightarrow\; u_0 = \ldots$

3) Robin boundary condition $u(0) + 2\frac{du}{dx}(0) = 5$: $\frac{u_0+u_1}{2} + 2\frac{u_1-u_0}{x_1-x_0} = 5 \;\Rightarrow\; u_0 = \ldots$

Substitute u_0 in the discretization for $i = 1$ given in c.2.

Note: although $x = 0$ is not exactly in the middle of x_0 and x_1 (due to the non-uniform grid), we did use here discretization formulas for the boundary conditions as if $x = 0$ is exactly in between the grid points.
This will also give second-order accurate discretization at the boundary:
we have that $(x_0 + x_1)/2 = (g(-h/2) + g(h/2))/2 \approx g(0)$ with an error $= (g(-h/2) + g(h/2) - 2g(0)/2 = h^2g''(0)/2 + O(h^3)$. This means that for the middle between x_0 and x_1 it holds that $(x_0 + x_1)/2 \approx 0$ with an error of $O(h^2)$.
if we have a second-order discretization in a point with distance $O(h^2)$ to $x = 0$ then the discretization will also be of second-order in $x = 0$.