Book: Bifurcation Analysis of Fluid Flows
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Adjoint $\mathcal{A}^*$ of $\mathcal{A}$ is s.t. $(v, \mathcal{A}u) = (\mathcal{A}^*v, u) \quad \forall u, v$

a.
1. use partial integration to show $(v, \frac{d}{dx}u) = v(1)u(1) - v(0)u(0) + (-\frac{d}{dx}v, u)$
2. Since we are without boundary conditions $u(0), u(1), v(0), v(1)$ are not known. Hence, the adjoint is $-\frac{d}{dx}$ on the domain and $+1$ at $x = 1$ (as can be seen from the term $+(vu)(1)$) and -1 at $x = 0$.

b.
1. use twice partial integration to show $(v, \frac{d^2}{dx^2}u) = (v\frac{du}{dx} - \frac{dv}{dx}u)\Big|_0^1 + (\frac{d^2}{dx^2}v, u)$. Hence the adjoint is $\frac{d^2}{dx^2}$ on the domain and the other term represents the adjoint on the boundary
2. rewrite as $(v, \frac{d^2}{dx^2}u) - v\frac{du}{dx}\Big|_0^1 = (\frac{d^2}{dx^2}v, u) - \frac{dv}{dx}u\Big|_0^1$ (on the left terms with v and on the right terms with u). On the left we have $(v, \mathcal{A}u)$ and on the right $(\mathcal{A}v, u)$ where both consist of a part on the domain and a part on the boundary.

c. Show (and give argumentations)
$(v, \int_0^x u(t)dt) = \ldots = \int_0^1 \int_0^x u(t)v(x)dtdx = \int_0^1 \int_t^1 u(t)v(x)dxdt = \int_0^1 \int_x^1 u(x)v(t)dtdx = (\int_x^1 v(t)dt, u)$.
(the third = comes from change of integration order (argue how this change works), the fourth = from interchanging names of the dummy variable). Conclude from this how the adjoint of the integral operator is defined.

d.
1. adjoint of $\frac{d}{dx}$ is $-\frac{d}{dx}$ if boundary conditions are such that $v(1)u(1) - v(0)u(0) = 0$, i.e. if boundary conditions are periodic (if $u(1) = u(0)$ and hence also $v(1) = v(0)$). Hence, in that case, skew-adjoint.
2. adjoint of $\frac{d^2}{dx^2}$ is $\frac{d^2}{dx^2}$ if boundary conditions are such that $(v\frac{du}{dx} - \frac{dv}{dx}u)\Big|_0^1 = 0$. This is the case is $u(0) = 0$ or $\frac{du}{dx}(0) = 0$, and $u(1) = 0$ or $\frac{du}{dx}(1) = 0$. Hence, in that case, self-adjoint.
3. Write $\int_0^1 u(t)dt = \int_0^x u(t)dt + \int_x^1 u(t)dt$. Use this to conclude that the integral operator is skew-adjoint if $\int_0^1 u(t)dt = 0$.