# Book: Bifurcation Analysis of Fluid Flows
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Consider $-u_{xx} = (\sin(2\pi(x - \frac{1}{4})))^2$ on $[0, 1]$ with periodic boundary conditions $u(0) = u(1)$. Compute approximate solution by using Galerkin projection on $\mathcal{V} = \text{span}\{v_1(x), v_2(x)\}$ with $v_1(x) = \sin(2\pi x)$, and $v_2(x) = \cos(2\pi x)$.

1. Define residual $r(u) = (\sin(2\pi(x - \frac{1}{4})))^2 - (-u_{xx})$
   Galerkin projection: find $u_{\mathcal{V}} \in \mathcal{V}$ s.t. $(v, r(u_{\mathcal{V}})) = 0 \;\; \forall v \in \mathcal{V}$
   Write $(v, r(u_{\mathcal{V}})) = 0$ as $(v, u''_{\mathcal{V}}) = (v, (\sin(2\pi(x - \frac{1}{4})))^2)$

2. Take $u_{\mathcal{V}} = a_1 v_1(x) + a_2 v_2(x)$, then $a_1, a_2$ can be computed from

$$\begin{pmatrix} (v_1, v_1'') & (v_1, v_2'') \\ (v_2, v_1'') & (v_2, v_2'') \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (v_1, \sin(2\pi(x - \frac{1}{4})))^2) \\ (v_2, \sin(2\pi(x - \frac{1}{4})))^2) \end{pmatrix}$$

   Take $v_1(x) = \sin(2\pi x), v_2(x) = \cos(2\pi x)$ and compute the matrix and right-hand side of this system by computing the inner products (i.e. by computing the integrals).

3. Solve the system, and then compute $u_{\mathcal{V}}(x) = a_1 v_1(x) + a_2 v_2(x)$