Book: Bifurcation Analysis of Fluid Flows
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a. Project $f(x) = e^x$ on $[0, 1]$ on $\mathcal{V} = \text{span}\{v_1(x), v_2(x), v_3(x)\}$.
Projection $f_\mathcal{V} = a_1 v_1(x) + a_2 v_2(x) + a_3 v_3(x)$ where $\vec{a} = (a_1, a_2, a_3)^T$ can be computed from $A\vec{a} = b$, where $A_{ij} = (v_i, v_j)$ and $b_i = (v_i, f)$. The inner product is defined as $(v_i, v_j) = \int_0^1 v_i(x)v_j(x)dx$.
Take $v_1(x) = 1, v_2(x) = x, v_3(x) = x^2$.

1. Using $v_i(x) = x^{i-1}$ you can show $(v_i, v_j) = \int_0^1 x^{i+j-2}dx$.
2. Then, show

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} e-1 \\ 1 \\ e-2 \end{pmatrix}$$

3. Solve the system, and then compute $f_\mathcal{V} = a_1 + a_2 x + a_3 x^2$

b. Analogous approach.
Now $f(x) = e^{\sin(\pi x)}$ on $[0, 1]$ and $v_1(x) = 1, v_2(x) = \sin(\pi x), v_3(x) = \cos(\pi x)$.

1. Show (you can use Mathematica or other software package)

$$\begin{pmatrix} 1 & \frac{2}{\pi} & 0 \\ \frac{2}{\pi} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1.97631 \\ 1.42854 \\ 0 \end{pmatrix}$$

2. Solve the system, and then compute $f_\mathcal{V} = a_1 + a_2 \sin(\pi x) + a_3 \cos(\pi x)$