Book: Bifurcation Analysis of Fluid Flows
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a. i. Consider $-\frac{d}{dx}(e^x\frac{du}{dx}) = f$.
What boundary values are allowed? At one boundary, lets say $x = 1$ we set a Dirichlet condition (needed to prove coercivity using Pointcaré), on the other boundary we consider Robin boundary condition $au(0) + b\frac{du}{dx}(0) = c$ and check what values of a and b we can take.
In weak form $a(v, u) = F(v)$ given by $(v, f-(-\frac{d}{dx}e^x\frac{du}{dx}))+v(0)(c-(au(0)+b\frac{du}{dx}(0))) = 0$ (we did not introduce an α to assure coercivity of $a(u, v)$ because we will choose the values a and b boundary conditions (since they are not given) such that coercivity can be assured).

$$\begin{aligned}
a(v,u) &= \int_0^1 -\frac{d}{dx}(e^x\frac{du}{dx})v\ dx + v(0)(au(0) + b\frac{du}{dx}(0)) \\
&= -e^x\frac{du}{dx}v\Big|_0^1 + \int_0^1 e^x\frac{du}{dx}\frac{dv}{dx}\ dx + v(0)(au(0) + b\frac{du}{dx}(0)) \\
F(v) &= (v, f) + v(0)c \\
\Rightarrow\ a(v,v) &= -e^x\frac{dv}{dx}v\Big|_0^1 + \int_0^1 e^x(\frac{dv}{dx})^2 dx + av(0)v(0) + b\frac{dv}{dx}(0)v(0) \\
&= (b+1)\frac{dv}{dx}(0)v(0) + \int_0^1 e^x(\frac{dv}{dx})^2 dx + av(0)^2
\end{aligned}$$

where we used $v(1) = 0$
For coercivity $a(v, v) \geq 0$, and hence all terms need to be non-negative. Hence we need to take $b = -1$ (to cancel the term $v(0)\frac{dv}{dx}(0)$ of which we do not know the sign), and $a > 0$. Indeed $a(v, v) = 0$ only if $v = 0$, since $a(v, v) = 0$ only if $\frac{dv}{dx} = 0$, which is the case if $v = c$ but since $v(1) = 0$ it only holds if $v = 0$.
Concluding: as boundary condition we can take $au(0) - \frac{du}{dx}(1) = c$ with $a > 0$.

ii. 1. Consider $-\frac{d^2u}{dx^2} - \frac{du}{dx} = e^{-x}f$. Weak form $a(v, u) = F(v)$ given by $(v, -\frac{d^2u}{dx^2} - \frac{du}{dx}) = (v, e^{-x}f)$.

$$\begin{aligned}
a(v,u) &= \int_0^1 -\frac{d^2u}{dx^2}v - \frac{du}{dx}v\ dx = -\frac{du}{dx}v\Big|_0^1 + \int_0^1 \frac{du}{dx}\frac{dv}{dx}\ dx - \int_0^1 \frac{du}{dx}v\ dx \\
\Rightarrow\ a(v,v) &= -\frac{dv}{dx}v\Big|_0^1 + \int_0^1 (\frac{dv}{dx})^2\ dx - \int_0^1 \frac{dv}{dx}v\ dx = -\frac{dv}{dx}v\Big|_0^1 + \int_0^1 (\frac{dv}{dx})^2\ dx - \frac{1}{2}v^2\Big|_0^1 \\
&= -\frac{dv}{dx}v\Big|_0^1 + \int_0^1 (\frac{dv}{dx})^2\ dx - \frac{1}{2}v(1)^2 + \frac{1}{2}v(0)^2
\end{aligned}$$

To get only non-negative terms we should at $x = 1$ prescribe $v(1) = 0$. In the next step we will look at the boundary condition at $x = 0$.

2. Consider a general boundary condition at $x = 0$, i.e. $au(0) + b\frac{du}{dx}(0) = c$ and check what values a and b can take.
   This boundary condition introduces an additional residual $r_2(u) = c - (au(0) + b\frac{du}{dx}(0))$, and consequently a term $v(0)(au(0) - b\frac{du}{dx}(0))$ in $a(u, v)$ and a term $cv(0)$ in $F(v)$ (we did not introduce an α to assure coercivity of $a(u, v)$ because we will choose the values a and b boundary conditions (since they are not given) such that coercivity can be assured).
   We get $a(v, u) = (v, -\frac{d^2u}{dx^2} - \frac{du}{dx}) + v(0)(au(0) + b\frac{du}{dx}(0))$. Hence, using partial integration,

$$a(v, v) = \frac{dv}{dx}(0)v(0) + \int_0^1 (\frac{dv}{dx})^2 \, dx + (a + \frac{1}{2}v(0)^2) + bv(0)\frac{dv}{dx}(0)$$

   where we used $v(1) = 0$.
   For coercivity $a(v, v) \geq 0$, and hence all terms need to be non-negative. Hence we need to take $b = -1$ (to cancel the term $v(0)\frac{dv}{dx}(0)$ of which we do not know the sign), and $a + \frac{1}{2} > 0$, i.e. $a > -\frac{1}{2}$. Indeed $a(v, v) = 0$ only if $v = 0$, since $a(v, v) = 0$ only if $\frac{dv}{dx} = 0$, which is the case if $v = c$ but since $v(1) = 0$ it only holds if $v = 0$.
   Concluding: as boundary condition we can take $au(0) - \frac{du}{dx}(1) = c$ with $a > -\frac{1}{2}$.

i. $\leftrightarrow$ ii. Equation (ii) is obtained from equation (i) by multiplying with a positive function (in this case e^{-x}). As a consequence of this multiplication a larger class of boundary conditions at $x = 0$ is allowed, i.e Robin condition $au(0) - \frac{du}{dx}(1) = c$ with $a > 0$ for (i) and $a > -\frac{1}{2}$ for (ii).

Instead of applying a Galerkin approach to (i) this comes down to applying a Petrov-Galerkin approach to (i), instead of having the search and test space both equal to $\mathcal{V}$ (Galerkin), it has different spaces (Petrov-Galerkin), the test space is now $e^{-x}\mathcal{V}$.

b. 1. Convection-diffusion equation $-\bar{u}u_x + \mu u_{xx} + f = 0$ can be written as $-u_{xx} + \frac{\bar{u}}{\mu}u_x = \frac{1}{\mu}f$ which is of the form (ii)

   Consider $-\frac{d}{dx}(e^{\alpha x}\frac{du}{dx}) = g(x)$ and show that this can be written as $-\alpha\frac{du}{dx} - \frac{d^2u}{dx^2} = e^{-\alpha x}g$

   When setting $e^{-\alpha x}g = \frac{1}{\mu}f$ and $-\alpha = \frac{\bar{u}}{\mu}$ one can show $e^{-\frac{\bar{u}}{\mu}x}(-u_{xx} + \frac{\bar{u}}{\mu}u_x) = e^{-\frac{\bar{u}}{\mu}x}\frac{1}{\mu}f$

   Hence, one can show $-\frac{d}{dx}(e^{-\frac{\bar{u}}{\mu}x}\frac{du}{dx}) = \frac{1}{\mu}e^{-\frac{\bar{u}}{\mu}x}f$ which is of the form (i)

2. Show $(v, -\frac{d}{dx}(e^{-\frac{\bar{u}}{\mu}x}\frac{du}{dx})) = -e^{-\frac{\bar{u}}{\mu}x}\frac{du}{dx}v\Big|_0^1 + \int_0^1 e^{-\frac{\bar{u}}{\mu}x}\frac{du}{dx}\frac{dv}{dx}dx$.

   Argue that this will result in $a(v, u) = \int_0^1 e^{-\frac{\bar{u}}{\mu}x}\frac{du}{dx}\frac{dv}{dx}dx$ and hence
   $a(v, v) \geq \min_{x\in[0,1]}(e^{-\frac{\bar{u}}{\mu}x})\int_0^1(\frac{dv}{dx})^2dx$.

   Argue that $a(v, v) \geq \min_{x\in[0,1]}(e^{-\frac{\bar{u}}{\mu}x})\min(\frac{1}{2}, \frac{1}{2L^2})\|v\|_{\mathcal{V}}$ (use Poincaré)

   Hence, the coercivity constant is $c = e^{-\frac{\bar{u}}{\mu}}\min(\frac{1}{2}, \frac{1}{2L^2})$ which $\to 0$ for $\frac{\bar{u}}{\mu} \to \infty$.

3. Coercivity constant c close to 0, means that $a(.,.)$ is coercive, but 'barely'. The problem will have a unique solution, but will be poorly conditioned, i.e. small perturbations in the problem might lead to large perturbations in the solution.