Book: Bifurcation Analysis of Fluid Flows
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Below the number between parenthesis refers to the number in the table of te boundary condition considered or, for the second table, the problem.

a.(1). From differential equation to weak form

Consider $-\frac{d^2u}{dx^2} + (1+x)u = x^2$ on $\Omega = [0,1]$

1. Boundary conditions $u(0) = u(1) = 0$
   Hence, we search solutions in $\mathcal{W} = \{u \in C^2[0,1] | u(0) = u(1) = 0\}$
2. Galerkin approach:
   Define residual $r(u) = x^2 - (-\frac{d^2u}{dx^2} + (1+x)u)$.
   find $u_{\mathcal{V}} \in \mathcal{V}$ s.t. $(v, r(u_{\mathcal{V}})) = 0 \;\; \forall v \in \mathcal{V}$, Define $\mathcal{V}$ later on.
   Write as $a(v,u) = F(v)$. Therefor, use partial integration and use $v(0) = v(1) = 0$ (because $u(0) = u(1) = 0$ and v is in the same space as u) to show that we can write

   $$(v, -\frac{d^2u}{dx^2} + (1+x)u) = (\frac{dv}{dx}\frac{du}{dx}) + (v, (1+x)u)$$

   Conclude that we can write the problem as find $u \in \mathcal{V}$ s.t. for all $v \in \mathcal{V}$

   $$a(v,u) = F(v) \text{ with } a(v,u) = (\frac{dv}{dx}, \frac{du}{dx}) + (v, (1+x)u), \quad F(v) = (v, x^2)$$
3. Since the weak form consists of first order derivative (and no longer second order) we can expand $\mathcal{W}$ to $\hat{\mathcal{W}} = \{u \in H^1[0,1] | u(0) = u(1) = 0\}$ Taking $\mathcal{V} = \hat{\mathcal{W}}$ gives

   $$\mathcal{V} = \{v \in H^1[0,1] | v(0) = v(1) = 0\}$$

c.(3) From weak form to differential equation

$a(v,u) = ((1+\cos(x))\frac{dv}{dx}, \frac{du}{dx}), \quad F(v) = (v, \exp(x))$

$\mathcal{V} = \{v \in H^1[0,1] \mid v(0) = v(1) = 0\}$

1. Determine differential equation: rewrite $a(v,u) = ((1+\cos(x))\frac{dv}{dx}, \frac{du}{dx})$ in the form $a(v,u) = (v, .)$ where the second argument of the inner product does not depend on v.

   Show using partial integration and $v(0) = v(1) = 0$ that we can write

   $a(v,u) = (v, -\sin(x)\frac{du}{dx} + (1+\cos(x))\frac{d^2u}{dx^2})$.

   Hence, the differential equation is $-\sin(x)\frac{du}{dx} + (1+\cos(x))\frac{d^2u}{dx^2} = \exp(x)$

2. Determine boundary conditions for u: since $u \in \mathcal{V}$, we have $u(0) = u(1) = 0$.

c.(4)
1. Determine differential equation: show using partial integration and $v(0) = v(1) = 0$ that we can write $a(u, v) = (v, (-\frac{du}{dx} - \frac{d^2u}{dx^2}))$. Hence, $-\frac{du}{dx} - \frac{d^2u}{dx^2} = \exp(x)$
2. Determine boundary conditions

d.a.(1) We have

$$a(v, u) = (\frac{dv}{dx}, \frac{du}{dx}) + (v, (1+x)u)$$
$$\Rightarrow \; a(u, u) = (\frac{du}{dx}, \frac{du}{dx}) + (u, (1+x)u) = \int_0^1 (\frac{du}{dx})^2 dx + \int_0^1 (1+x)u^2 dx \geq 0$$

because $(\frac{du}{dx})^2 \geq 0, u^2 \geq 0$ and $1 + x > 0$ for $x \in [0, 1]$. Since $a(u, u) = 0$ if $\frac{du}{dx} = 0$ and $u = 0$, we see $a(u, u) = 0$ only if $u = 0$.
Concluding: $a(., .)$ is positive definite.

d.c.(3) Argue that $a(u, u) = \int_0^1 (1 + \cos(x))(\frac{du}{dx})^2$ is ≥ 0, and that $a(u, u) = 0$ if $\frac{du}{dx} = 0$, hence if $u(x) = c$, but since the boundary conditions are such that $u(0) = 0, u(1) = 0$ it must hold that $u(x) = 0$ (hence $c = 0$). Hence, $a(u, u) = 0$ only if $u = 0$. Conclude that $a(., .)$ is positive definite.

d.c.(4) Show that $a(u, u) = \int_0^1 u\frac{du}{dx}dx + (\frac{du}{dx})^2 dx$. The second integral is ≥ 0, but the first integral (and hence $a(u, u)$) is not necessarily larger than 0. Hene $a(., .)$ not positive definite.

e.a.(1) From d.a.(1) we know $a(., .)$ is positive definite. Now show $a(v, u) = a(u, v)$. Then, conclude $a(., .)$ is symmetric positive definite and hence we can write problem as minimization: find $u_{\mathcal{V}} \in \mathcal{V}$ s.t. $u_{\mathcal{V}} = \text{argmin}_{v \in \mathcal{V}}(a(v, v) - 2F(v))$

e.c.(3) Reasoning similar as e.a.(1)

e.c.(4) From d.c.(4) we know $a(., .)$ is not positive definite. Hence, we can not write problem as minimization problem.

a+d. Consider $-\frac{d^2u}{dx^2} + (1 + x)u = x^2$ on $\Omega = [0, 1]$

a+d.(2) Boundary conditions $u(0) = 1, \quad u(1) = 5$.

1. write $u(x) = \bar{u}(x) + \tilde{u}(x)$ s.t. $\bar{u}(x)$ is fixed and $\tilde{u}(0) = 0, \quad \tilde{u}(1) = 0$.
   Hence, take $\bar{u}(x)$ s.t. $\bar{u}(0) = 1, \quad \bar{u}(1) = 5 \; \Rightarrow \bar{u} = 1 + 4x$.
2. Give problem for $\tilde{u}$:
   $-\frac{d^2\tilde{u}}{dx^2} + (1 + x)\tilde{u} = x^2 - (1 + x)(1 + 4x)$ with $\tilde{u}(0) = 0, \quad \tilde{u}(1) = 0$.
3. solve using approach analogously to the one in part a.(1)

a+d.(3) Boundary conditions $\frac{du}{dx}(0) = 3$, $u(1) = 0$

1. Argue that $\mathcal{V} = \{v \in H^1[0, 1] \;\Big|\; v(1) = 0\}$.
   Note: $\mathcal{V}$ only has boundary conditions where u has Dirichlet boundary conditions, if u has Neumann or Robin boundary conditions at a boundary, $\mathcal{V}$ does not impose a boundary condition at that boundary on its elements.
2. Consider residual $r_1(u) = x^2 - (-\frac{d^2u}{dx^2} + (1 + x)u)$ and $r_2(u) = 3 - \frac{du}{dx}(0)$.
   Galerkin approach: find $u \in \mathcal{V}$ s.t. $(v, r_1(u)) + \alpha v(0) r_2(u) = 0 \;\; \forall v \in \mathcal{V}$

3. show this can be written as $a(v,u) = F(v)$ with $a(v,u) = v(0)\frac{du}{dx}(0) + \int_0^1 \frac{dv}{dx}\frac{du}{dx}dx + \int_0^1 (1+x)vudx + \alpha v(0)\frac{du}{dx}(0)$ and $F(v) = (v, x^2) + \alpha 3v(0)$
4. Now choose α such that $a(.,.)$ will be positive definite, and can possibly be coercive. Consider $a(v,v) = v(0)\frac{dv}{dx}(0) + \int_0^1 (\frac{dv}{dx})^2 dx + \int_0^1 (1+x)v^2 dx + \alpha v(0)\frac{dv}{dx}(0)$
The term $v(0)\frac{dv}{dx}(0)$ of which the sign is not known should cancel, i.e. choose $\alpha = -1$. Then $a(v,v) = \int_0^1 (\frac{dv}{dx})^2 dx + \int_0^1 (1+x)v^2 dx$. Which is ≥ 0 since $(\frac{dv}{dx})^2 \geq 0$ and $(1+x)v^2 \geq 0$ (the latter because $1+x > 0$ for $x \in [0,1]$). Where $= 0$ only if $\frac{dv}{dx} = 0$ and $v = 0$. Hence $a(v,v) \geq 0$ and $a(v,v) = 0$ only if $v = 0$.
5. Hence Galerkin approach: find $u \in \mathcal{V}$ s.t. $a(v,u) = F(v)$ with $a(v,u) = \int_0^1 \frac{dv}{dx}\frac{du}{dx}dx + \int_0^1 (1+x)vudx$ and $F(v) = (v, x^2) - 3v(0) \quad \forall v \in \mathcal{V}$

a+d.(4) Boundary conditions $\frac{du}{dx}(0) = 3$, $u(1) = 5$

1. we need a zero boundary condition, define $u(x) = 5 + \tilde{u}(x)$ and solve problem in terms of $\tilde{u}$.
2. Show $\frac{du}{dx}(0) = 3, \quad u(1) = 5 \ \Rightarrow \frac{d\tilde{u}}{dx}(0) = 3, \quad \tilde{u}(1) = 0$ and $-\frac{d^2u}{dx^2} + (1+x)u = x^2 \ \Rightarrow \ -\frac{d^2\tilde{u}}{dx^2} + (1+x)u = x^2 - 5(1+x)$.
3. Argue that $\mathcal{V} = \{v \in H^1[0,1] \ \Big| \ v(1) = 0\}$.
4. Now apply Galerkin to equations for $\tilde{u}$, using residuals $r_1(u) = x^2 - 5(1+x) - (-\frac{d^2u}{dx^2} + (1+x)u)$ and $r_2(u) = 3 - \frac{du}{dx}(0)$ (note that we use still u here for simplicity of notation instead of $\tilde{u}$).
5. show $\alpha = -1$ and show that the problem can be written as $a(v,u) = F(v)$ with $a(v,u) = \int_0^1 \frac{dv}{dx}\frac{du}{dx}dx + \int_0^1 (1+x)vudx$ and $F(v) = (v, x^2 - 5(1+x)) - 3v(0)$.
6. show $a(v,u)$ positive definite

a+d.(5) Boundary conditions $\frac{du}{dx}(0) + 9u(0) = 3$, $u(1) = 0$

1. Argue that $\mathcal{V} = \{v \in H^1[0,1] \ \Big| \ v(1) = 0\}$.
2. Apply Galerking approach using residuals $r_1(u) = x^2 - (-\frac{d^2u}{dx^2} + (1+x)u)$ and $r_2(u) = 3 - (\frac{du}{dx}(0) + 9u(0))$
3. Show it can be written as $a(v,u) = F(v)$ with $a(v,u) = v(0)\frac{du}{dx}(0) + \int_0^1 \frac{dv}{dx}\frac{du}{dx}dx + \int_0^1 (1+x)vudx + \alpha v(0)(\frac{du}{dx}(0) + 9u(0))$ and $F(v) = (v, x^2) + \alpha 3v(0)$.
4. consider $a(v,v)$ and show that we should take $\alpha = -1$ and that $a(u,v) = \int_0^1 \frac{dv}{dx}\frac{du}{dx}dx + \int_0^1 (1+x)vudx - 9v(0)u(0)$ and $F(v) = (v, x^2) - 3v(0)$.
But conclude that $a(.,.)$ is not positive definite and hence not coercive, because of the term $-9v(0)u(0)$.
Hence the problem does not satisfy the Lax-Milgram Theorem, this means that we are not able to conclude whether the problem is well-posed or not (i.e. whether it has a unique solution or not)

a+d.(6) Boundary conditions $\frac{du}{dx}(0) - 9u(0) = 3$, $u(1) = 0$

1. almost similar to part a.(4) except that it will result in $a(u,v) = \int_0^1 \frac{dv}{dx}\frac{du}{dx}dx + \int_0^1 (1+x)vudx + 9v(0)u(0)$.
2. argue that now $a(v,u)$ is positive definite

b. Consider $-\frac{d}{dx}(\exp(x)\frac{du}{dx}) = \sin(x)$ on $\Omega = [-1, 0]$
Boundary conditions $u(-1) = 0, \quad \frac{du}{dx}(0) = 5$.

1. Argue that $\mathcal{V} = \{v \in H^1[0,1] \,\Big|\, v(-1) = 0\}$.
2. Apply Galerking approach using residuals
$r_1(u) = \sin(x) - (-\frac{d}{dx}(\exp(x)\frac{du}{dx})$ and $r_2(u) = 5 - \frac{du}{dx}(0)$
3. show that we should choose $\alpha = 1$ and that
$a(v,u) = \int_{-1}^{0} \frac{dv}{dx}\exp(x)\frac{du}{dx}dx$ and $F(v) = (v, \sin(x)) + 5v(0)$
4. show that $a(v,v) \geq 0$ and $a(v,v) = 0$ only if $v = 0$. For the latter: $a(v,v) = 0$ only if $\frac{dv}{dx} = 0$ and hence if $v = c$, but since in $\mathcal{V}$ we have $v(-1) = 0$ this results in $c = 0$ and hence $a(v,v) = 0$ only if $v = 0$.
Hence $a(.,.)$ positive definite

c+d. Compare to Exercise 1.15 c+d.3 and c+d.4 in Hints Tutorial 1.

c+d.(1) Given $a(v,u) = 5(\frac{dv}{dx}, \frac{du}{dx}) + 3v(0)u(0)$, $F(v) = (v, \cos(x)) - 5v(0)$ and $\mathcal{V} = \{v \in H^1[0,1] \,\Big|\, v(1) = 0\}$

1. Write in the form $(v, \ldots) + v(0)\ldots$ where the $\ldots$ depend on u and not on v. Then the second argument of the inner product $(v, \ldots)$ holds the differential equation and the second term in $v(0)\ldots$ the boundary condition at $x = 0$
Show that we can rewrite $a(v,u) = F(v)$ as $\int_0^1 v(-5\frac{d^2u}{dx^2} - \cos x)dx + v(0)(3u(0) - 5\frac{du}{dx}(0) + 5) = 0$ for arbitrary $v \in \mathcal{V}$
2. argue that we get the differential equation $-5\frac{d^2u}{dx^2} - \cos x = 0$ with boundary conditions $u(1) = 0$ and $3u(0) - 5\frac{du}{dx}(0) + 5 = 0$. Or similar: $-5\frac{d^2u}{dx^2} = \cos x$ with boundary conditions $u(1) = 0$ and $3u(0) - 5\frac{du}{dx}(0) = -5$.
3. show $a(v,v) = 5\int_0^1 (\frac{dv}{dx})^2 dx + 3v(0)^2$ and argue that $a(v,v) \geq 0$ and $a(v,v) = 0$ only if $\frac{dv}{dx} = 0$, i.e. $v = c$, and $v(0) = 0$, and hence only if $v = 0$. Conclude: $a(u,v)$ positive definite.

c+d.(2) Given $a(v,u) = 5(\frac{dv}{dx}, \frac{du}{dx}) - 3v(0)u(0)$, $F(v) = (v, \cos(x)) - 5v(0)$ and $\mathcal{V} = \{v \in H^1[0,1] \,\Big|\, v(1) = 0\}$

1. similar to part c+d.(2)
2. argue that we get the differential equation $-5\frac{d^2u}{dx^2} - \cos x = 0$ with boundary conditions $u(1) = 0$ and $-3u(0) - 5\frac{du}{dx}(0) + 5 = 0$. Or similar: $-5\frac{d^2u}{dx^2} = \cos x$ with boundary conditions $u(1) = 0$ and $-3u(0) - 5\frac{du}{dx}(0) = -5$.
3. show that $a(u,v)$ is not positive definite. Hence, the conditions of Lax-Milgram are not satisfied and hence the problem might not be well-posed, i.e. the solution might not exist or might not be unique.

e. For all problems for which $a(v,u)$ is symmetric and positive definite we can write the problem as a minimization problem: find $u_{\mathcal{V}} = \text{argmin}_{v \in \mathcal{V}}(a(v,v) - 2F(v))$.
In all of our cases $a(v,u)$ is symmetric (show!), but as we have seen not in all cases it is positive definite.