Book: Bifurcation Analysis of Fluid Flows
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Chapter: 3, Exercise: 3.12
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Version: 1

a.
1. Since A is symmetric, its eigenvalues are real and the matrix is diagonalizable, i.e. its eigenvectors form an orthonormal basis for $\mathbb{R}^n$. Since A is positive definite its eigenvalues are larger than 0.
2. Say A has eigenvalues $0 < \lambda_1 \leq ... \leq \lambda_n$ and corresponding orthonormal eigenvectors v_i then we can write $x = \alpha_1 v_1 + \ldots + \alpha_n v_n$ with some constants α_i which gives

$$\begin{aligned} x^T A x &= (\sum \alpha_j v_j)^T A (\sum \alpha_i v_i) \\ &= ... = \sum \sum \alpha_j \alpha_i \lambda_i v_j^T v_i \\ &= \lambda_1 \alpha_1^2 + \ldots + \lambda_n \alpha_n^2 \\ &\geq \lambda_1 \alpha_1^2 + \ldots + \lambda_1 \alpha_n^2 \\ &= \lambda_1 \|x\|^2 \end{aligned}$$

where we used $v_j^T v_i = 0$ if $i \neq j$, $v_i^T v_i = 1$ (because v_i orthonormal), and consequently $\|x\|^2 = x^T x = \sum \alpha_i^2$.

b.
1. Assume A is singular, $\exists\ x \neq 0$ s.t. $Ax = 0$, and hence $x^T Ax = 0$. This contradicts with the given that $x^T Ax \geq x^T x$ for some c. Hence A non-singular.
2. If A is non-singular, the problem $Ax = b$ has a unique solution.

Remark 1:
The above agrees with Lax-Milgram. If the matrix A satisfies coercivity, it is non-singular and $Ax = b$ has a unique solution (problem is well-posed).
Remark 2:
Part c. and d. do not look at the whole space $\mathbb{R}^n$ but at a subspace $\mathcal{V} \subset \mathbb{R}^n$ (note that the solution of $Ax = b$ is not necessarily in $\mathcal{V}$). Galerkin approach: find $x_\mathcal{V} \in \mathcal{V}$ such that $(v, Ax_\mathcal{V} - b) = 0\ \ \forall\, v \in \mathcal{V} \ \ \Rightarrow \ \ v^T A x_\mathcal{V} = v^T b\ \ \forall\, v \in \mathcal{V}$. Let V be the matrix which columns are the basis vectors of $\mathcal{V}$ (if $\dim(\mathcal{V}) = m$ then V is an $n \times m$ matrix), then the Galerkin approach can be written as: find $x_\mathcal{V} \in \mathcal{V}$ such that $V^T A x_\mathcal{V} = V^T b$. Since $x_\mathcal{V} \in \mathcal{V}$ we know $\exists\ \hat{x} \in \mathbb{R}^m$ s.t. $x_\mathcal{V} = V\hat{x}$. This results in: find $\hat{x} \in \mathbb{R}^m$ s.t. $V^T AV\hat{x} = V^T b$ where $V^T AV$ (an $m \times m$ matrix).
Part c. shows that when the matrix $V^T AV$ is positive definite (and hence coercive, finite dimensional case), the problem $V^T AV\hat{x} = V^T b$ is well-posed, has a unique solution. Part d. shows that when the matrix $V^T AV$ is not positive definite (and hence not coercive), the problem $V^T AV\hat{x} = V^T b$ need not be well-posed ($V^T AV$ singular and hence not a unique solution).

c. Note: A does not have to be symmetric here!
Assume A an $n \times n$ matrix, positive definite. Hence $(x, Ax) > 0 \;\; \forall \; x \neq 0$ where $x \in \mathbb{R}^n$, and consequently $(v, Av) > 0 \;\; \forall \; v \neq 0$ where $v \in \mathcal{V} \subset \mathbb{R}^n$.
If A would be singular on $\mathcal{V}$ then $\exists \; v \neq 0$ s.t. $Av = 0$, and hence $v^T Av = 0$. This contradicts with the given that $v^T Av > 0$. Hence, the Galerkin approximation of A is non-singular.

d.
1. $v^T Av = 0 \Rightarrow v = \alpha(1 \;\; 1)^T$, i.e. $\mathcal{V} = \text{span}\{(1 \;\; 1)^T\}$. Then $V^T AV = 0$, and hence is a singular matrix.
2. For Petrov-Galerkin: the search space $\mathcal{V}$ is not the same as the test space $\mathcal{W}$, i.e. find $x_{\mathcal{V}} \in \mathcal{V}$ such that $(w, Ax_{\mathcal{V}} - b) = 0 \;\; \forall \; w \in \mathcal{W}$. Taking $\mathcal{W} = A\mathcal{V}$ results in $(AV)^T AV\hat{x} = (AV)^T b$. The Petrov-Galerkin approximation of A is $V^T A^T AV$, which can be considered as Galerkin approximation of $A^T A$, which is a positive definite matrix and hence by part c. is non singular.