Book: Bifurcation Analysis of Fluid Flows
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Chapter: 3, Exercise: 3.11
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Version: 1

$J(v) = \frac{1}{2}a(v,v) - F(v)$ (note: $a(.,.)$ is bi-linear and $F(.)$ is linear).
Show that when $a(.,.)$ symmetric $J(u+\epsilon w) = J(u) + \epsilon(a(w,u) - F(w)) + \frac{1}{2}\epsilon^2 a(w,w)$:

1. show $a(u+\epsilon w, u+\epsilon w) = a(u,u) + 2\epsilon a(w,u) + \epsilon^2 a(w,w)$, using bi-linearity and symmetry of $a(.,.)$ (denote in your computation when you are using each of these properties)
2. $F(u+\epsilon w) = F(u) + \epsilon F(w)$ because of linearity.
3. rewrite $J(u+\epsilon w)$ using the above

What does it mean if u is a stationary point of $J(u)$?

1. u stationary point if $\frac{d}{d\epsilon}J(u+\epsilon w)\Big|_{\epsilon=0} = 0 \quad \forall w$ (directional derivative)
2. show that this results in: u such that $a(u,w) = F(w) \quad \forall w$

Remark:
the conclusion from Exercise 1.11 is that solving $\text{argmin}_u J(u) = \text{argmin}_u(\frac{1}{2}a(u,u) - F(u))$ is equivalent to solving $a(u,w) = F(w) \quad \forall w$ (where we saw the first one when solving $\mathcal{A}u = f$ using minimization and the second one when using Galerkin approximation).